ISRAEL JOURNAL OF MATHEMATICS 76 (1991), 161–182

ALMOST SURE CONVERGENCE THEOREMS IN VON NEUMANN ALGEBRAS

BY

M. S. GOLDSTEIN*

Department of Mathematics,
Ben-Gurion University of the Negev, Beer Sheva, Israel

AND

G. YA. GRABARNIK**

Institute of Mathematics,
The Hebrew University of Jerusalem, Jerusalem, Israel

ABSTRACT

The subadditive sequences of operators which belong to a von Neumann algebra with a faithful normal state and a given positive linear kernel are considered. We prove the almost sure convergence in Egorov's sense for such sequences.

Introduction

This paper is devoted to a presentation of some results concerning strong limit theorems in non-commutative probability which the authors proved in recent years. The first results in this field were obtained by Sinai and Anshelevich [20] and Lance [17], who showed almost sure convergence in Egorov's sense [20], [17], [19] in an ergodic theorem for transformations of von Neumann algebras (earlier Kovacs and Szücs [15] showed mean convergence in this case). During the last 10 years numerous results were proved, the main part of which were given in R. Jajte's monograph [12]. The first matter we consider in this paper is as follows. Let $\{x_n\}$ be a superadditive sequence, i.e. $x_{n+m} \geq x_n + \alpha^n(x_m)$ where α is a positive linear kernel in $M, \rho \circ \alpha = \rho$ (see [12]) and the number

* Partially supported by the Ministry of Science and Ministry of Absorption.
** Partially sponsored by a grant from the Edmund Landau Center for research in Mathematical Analysis, supported by the Minerva Foundation (Germany).
Received September 13, 1990 and in revised form September 15, 1991

sequence $\rho(n^{-1}x_n)$ is bounded. It is necessary to consider almost sure and mean convergence of $n^{-1}x_n$. This question is solved in section 1 under the additional condition $\sup_n n^{-1}||x_n|| < +\infty$. Note that in the case when the state ρ is a trace, this result was proved earlier by Jajte, but in the general case the research of the sequence $n^{-1}x_n$ is more difficult.

In section 2 we consider convergence of supermartingales, i.e. sequences $\{x_n\} \subset M$ of selfadjoint operators satisfying the condition

$$\varphi_n(x_n) = x_n, \quad \varphi_n(x_{n+1}) \geq x_n, n = 1, 2, \dots$$

where φ_n is an expectation from M on some von Neumann subalgebra M_n with respect to ρ and also $M_1 \subset M_2 \subset \cdots \subset M_n \subset \cdots$. Under natural conditions we obtain mean and almost sure convergence. The proof of this result is like that of the superadditive ergodic theorem from section 1. When the state ρ is a trace this result has been previously obtained by Cuculescu [5] (see also [2]) but the method of these works does not transfer to the case of a general state.

In section 3 we consider majorant convergence in ergodic theorems for transformations of von Neumann algebras. We assume the state ρ is a trace and prove for a selfadjoint operator X affiliated with M and $X \in L_p(M, \rho)$, $p > 1$ (see [24]) that there exists $\{Y_n\}_{n=1}^{\infty} \subset L_{p-\epsilon}(M, \rho)$, $\epsilon > 0$ (no matter how small) such that

$$-Y_n \leq \sigma_n(X) - (\hat{X}) \leq Y_n, \quad n = 1, 2, \dots, \quad \text{where} \quad \sigma_n(X) = n^{-1} \sum_{\kappa=0}^{n-1} \alpha^{\kappa}(X)$$

(see notation above), and

$$\hat{X} = \lim_{n \to \infty} \sigma_n(X).$$

From there it is easy to see the ergodic theorem for the average

$$n_1^{-1} \cdot n_2^{-1} \cdot \dots \cdot n_{\kappa}^{-1} \sum_{i_1=0}^{n_1-1} \dots \sum_{i_{\kappa}=0}^{n_{\kappa}-1} \alpha_1^{i_1} \dots \alpha_{\kappa}^{i_{\kappa}}(X)$$

where $\alpha_1, \dots, \alpha_{\kappa}$ are positive kernels in $M, \rho \circ \alpha_j \leq \rho, j = 1, 2, \dots, \kappa$.

The authors are grateful to the reviewer, who points out an elegant work of Dang-Ngoc [7], in which almost sure convergence for bounded martingales is proved.

1. Superadditive Ergodic Theorem

Let M be a von Neumann algebra acting on the Hilbert space H and let ρ be a faithful normal state on M defined by the separating cyclic vector ξ_0. Let α be a linear mapping from M into M satisfying the next conditions:

$$(1) \qquad \alpha M_+ \subset M_+, \qquad \alpha(\mathbf{1}) = \mathbf{1}, \qquad \rho(\alpha(x)) = \rho(x)$$

where M_+ is the set of positive elements from M, $\mathbf{1}$ is the identity of $M, x \in M$.

Denote the commutant of M by M' and denote the space of linear continuous (normal) forms acting on M' by $M'^*(M'_*)$ [8]. Every selfadjoint operator $x \in M$ is associated with a Hermitian normal functional ω_x acting on M', where $\omega_x(B) = (xB\xi_0, \xi_0)$ for all $B \in M'$.

We shall denote the norm of the functional $\varphi \in M'^*$ by $||\varphi||_1$. For $x = x^* \in M$ put $||x||_1 = ||\omega_x||_1$; it should be noted that $||x||_1 = \sup|(xB\xi_0, \xi_0)|$, where $B^* = B \in M', -\mathbf{1} \leq B \leq \mathbf{1}$, and therefore $||\alpha(x)||_1 \leq ||x||_1$.

We shall denote the norm of the operator x by $||x||_\infty$ and the norm of the vector $x\xi_0$ by $||x||_2$.

We shall denote $1/k\Sigma_{i=0}^{\kappa-1}T^iY$ by $\sigma_\kappa(T;Y)$ where T is a linear operator acting on the Banach space $L, Y \in L$. If it does not lead to a misunderstanding we omit the term T and write $\sigma_\kappa(Y)$. We say that the sequence $\{Y_n\}_{n=1}^\infty \subset M$ converges almost surely (a.s.) to $Y_0 \in M$ if for every $\epsilon > 0$ there exists a projection $E \in M$ such that $\rho(\mathbf{1} - E) < \epsilon, \lim_{n\to\infty} ||E(Y_n - Y_0)E||_\infty = 0$ [25]. We say that a sequence $\{x_n\}_{n=1}^\infty \subset M$ of selfadjoint operators is superadditive if there exists a linear map α such that (1) holds and

$$(2) \qquad x_{n+m} \leq x_n + \alpha^n x_m, \qquad \text{where } n, m = 1, 2, \ldots.$$

THEOREM 1.1 (see [12], [16], [10])): *Let $\{x_n\}_{n=1}^\infty \in M$ be a superadditive sequence and*

$$(3) \qquad \sup_{n\geq 1} ||x_n/n||_\infty = C < +\infty.$$

Then there exists a selfadjoint operator $x_0 \in M$ such that

$$\alpha x_0 = x_0, \quad \lim_{n\to\infty} ||x_n/n - x_0||_1 = 0;$$

x_n/n converges a.s. to x_0.

In order to begin the proof let us state the following lemmas:

LEMMA 1.2 ([9]): *Let α be a linear map from M to M satisfying (1). Then there exists a linear map $\alpha' : M' \to M'$ such that*
(i) $\alpha'(M'_+) \subset M'_+, \alpha'(\mathbf{1}) = \mathbf{1}, (\alpha'(B)\xi_0, \xi_0) = (B\xi_0, \xi_0)$ *for every* $B \in M'_+$.
(ii) $(\alpha(x)B_{\xi_0,\xi_0}) = (x\alpha'(B)\xi_0, \xi_0)$ *for all* $x \in M, B \in M'$.

Let α' be the operator acting on M' constructed in Lemma 1.2 from α. Then $((\alpha')^*(\omega_x))(B) = \omega_{\alpha(x)}(B)$ where $x = x^* \in M, B \in M'$ and the operator $(\alpha')^*$ transforms $(M')_*^+$ into $(M')_*^+$.

LEMMA 1.3 (see [12]): *Let $\{w_n\}_{n=1}^{\infty} \subset M$ be a superadditive sequence and*

$$y_n = 1/m \sum_{\kappa=1}^{m} (w_\kappa - \alpha w_{\kappa-1}), \qquad w_0 = 0, \quad m = 1, 2, \ldots.$$

There exists a sequence $\{z_n\}_{n=1}^{\infty} \subset M_+$ such that

$$n\sigma_n(\alpha, y_m) \geq w_n - m^{-1} z_n, \qquad 1 \leq n \leq m \tag{4}$$

and $\sup_{m\geq 1} ||y_m||_1 < \infty$.

LEMMA 1.4 (see [14], [12]): *Let $\{x_n\}_{n=1}^{\infty} \subset M_+$ be a superadditive sequence. There exists a positive normal functional $\bar{\omega}$ on M' such that*

$$||\bar{\omega}||_1 = \gamma = \lim_{n\to\infty} p(x_n/n), \quad \sigma((\alpha')^*, \bar{\omega}) \geq n^{-1}\omega_{x_n}, \qquad n = 1, 2, \ldots.$$

Proof: Let $\{y_n\}_{n=1}^{\infty}, \{z_n\}_{n=1}^{\infty}$ be constructed by Lemma 1.3 from the sequence $\{x_n\}_{n=1}^{\infty}$. The functionals ω_{y_n} are uniformly bounded and have an accumulation point ν_0 in the $\sigma((M')^*, M')$ topology. It follows from inequality (4) that $\sigma_n((\alpha')^*, \nu) \geq n^{-1}\omega_{x_n} \geq 0$. Then

$$||\nu_0|| = \nu_0(1) = \lim_s [m_s^{-1} \sum_{k=1}^{m_s} (\omega_{x_\kappa} - \omega_{\alpha x_{\kappa-1}})(1)] = \lim_s m_s^{-1} \rho(x_{m_s}) = \gamma,$$

where $\omega_{y_{m_s}}$ converges to ν_0. From the uniqueness of Takesaki's decomposition [22, p. 127] we have

$$(((\alpha')^*)^\kappa(\nu))_n = (\alpha')^*((((\alpha')^*)^{\kappa-1}(\nu_0))_n) + \nu_{n,\kappa},$$

where $\nu_{n,\kappa} = ((\alpha')^*((((\alpha')^*)^{\kappa-1}(\nu_0))_s))_n, \nu_{n,0} = (\nu_0)_n$ and $(\nu)_n((\nu)_s)$ are the normal (singular) parts of ν. Then

$$(((\alpha')^*)^\kappa(\nu_0))_n(1) = \sum_{i=0}^{\kappa} ||\nu_{n,i}||_1 \leq \gamma \tag{5}$$

and the series $\bar{\omega} = \Sigma_{i=0}^{\infty} \nu_{n,i}$ converges in the norm $\|\cdot\|_1$. Moreover

$$(6) \qquad n\sigma_n((\alpha')^*, \bar{\omega}) \geq \sum_{i=0}^{n-1} \sum_{\ell=0}^{i} ((\alpha')^*)^{i-l}(\nu_{n,\ell}) = \sum_{i=0}^{n-1} (((\alpha')^*)^i(\nu_0))_n \geq \omega_{x_n}.$$

From (6) and (5) it follows that $\|\bar{\omega}\| = \gamma$.

LEMMA 1.5 (see [6]): *Let ν be a normal Hermitian functional on M'. Then $\sigma_\kappa((\alpha')^*, \nu) \to \tilde{\nu}$ in $\|\cdot\|_1$, where $\tilde{\nu} = E\nu$ and E is the projection on the $(\alpha')^*$-invariant points in M'_* such that the range of the complementary projection is the closure of $\{(I - (\alpha')^*)(M'_*)\}$.*

Proof: Let K be the completion of the real linear space of all selfadjoint elements of M under the norm $\|\cdot\|_2$, and let $\tilde{K}$ be the complexification of K. From Kadison's inequality [13] $(\alpha x)^2 \leq \alpha(x^2)$, it follows that the unique extension of α on $\tilde{K}$ is a contraction in $\tilde{K}$. Since $\tilde{K}$ is reflexive, it follows from Corollary 8.5.4 [6] that

$$\sigma_n((\alpha')^*, \omega_x) \overset{\|\cdot\|}{\to} \omega_{Ex} \quad \text{when } n \to \infty,$$

and from inequality $\|x\|_1 \leq \|x\|_2$ correct for $x = x^* \in M$ it follows that

$$\sigma_n(\alpha, x) \overset{\|\cdot\|_1}{\to} E_x.$$

Corollaries 2 and 3 (8.5 [6]) finish the proof.

Proof of Theorem 1.1 (Norm $\|\cdot\|_1$ Convergence)*:* The sequence

$$\{x_n - n\sigma_n(\alpha, x_1)\}_{n=1}^{\infty}$$

is positive and superadditive.

By Lemma 1.4 there exists a normal Hermitian functional $\bar{\omega}$ such that

$$\sigma_n((\alpha')^*, \bar{\omega}) \geq n^{-1}\omega_{x_n} - \sigma_n((\alpha')^*, \omega_{x_1}) \quad \text{and} \quad \|\bar{\omega}\|_1 = \lim_{n\to\infty} \rho(x_n/n) - \rho(x_1).$$

Let $\hat{\bar{\omega}}$ be a limit of $\sigma_n((\alpha')^*, \bar{\omega})$ in the $\|\cdot\|_1$ norm. By Lemmas 1.4 and 1.5 we have

$$\begin{aligned} \|\omega_{x_{n/n}} - \hat{\bar{\omega}} + \omega_{\hat{x}_1}\|_1 \leq & \|\omega_{x_{n/n}} - \sigma_n((\alpha')^*, \omega_{x_1}) - \sigma_n((\alpha')^*, \bar{\omega})\|_1 \\ & + \|\sigma_n((\alpha')^*, \bar{\omega} - \hat{\bar{\omega}})\|_1 + \|\sigma_n((\alpha')^*, \omega_{x_1})\|_1 \to 0. \end{aligned}$$

From (3) it follows that $0 \leq \hat{\bar{\omega}} \leq 2c_0\omega_1$. By theorem I.4.5 [8] there exists $\hat{x}_0 \in M$ such that $\hat{\bar{\omega}} = \omega_{\hat{x}_0}$ or $\|x_{n/n} - \hat{x}_0 - \hat{x}_1\|_1 \to 0$ when $n \to \infty$.

Let us prove a.s. convergence.

THEOREM 1.6 (see [9]): *Let $\{A_n\}_{n=1}^N$ be a finite set of selfadjoint operators from M, $\{\varepsilon_n\}_{n=1}^N$ a finite set of positive numbers. If $\Sigma_{n=1}^N \epsilon_n^{-1}||A_n||_1 < 1/2$ then there exists a projection $E_N \in M$ with $\rho(E_N) \geq 1 - \Sigma_{n=1}^N \epsilon_n^{-1}||A_n||_1$ and such that*

$$||E_N \sigma_m(A_n) E_N||_\infty \leq \epsilon_n \quad \text{for } m, n = 1, 2, \ldots, N.$$

In the proof of the theorem we use finiteness of this set of selfadjoint operators. We also note that in this theorem A_n need not be positive as in [9].

LEMMA 1.7: *Let $\{w_n\}_{n=1}^\infty$ be a superadditive sequence, $\{y_s\}_{s=1}^\infty$ and $\{z_n\}_{n=1}^\infty$ the sequences of positive operators constructed by Lemma 1.3. Then*

$$w_\kappa \geq (\kappa - t)\sigma_{\kappa-t}(w_t/t), \text{ where } 1 \leq t \leq \kappa; \tag{7}$$

$$n\sigma_n(w_t/t) \geq (n-\kappa)\sigma_{n-\kappa}(\sigma_\kappa(w_t/t)), \text{ where } n \geq \kappa \geq 1; \tag{8}$$

$$\begin{aligned} &n\sigma_n(y_s - w_t/t) + \sum_{i=n-t}^{n-1} \alpha^i(w_t/t) + s^{-1} \cdot z_n \\ &\geq w_n - n\sigma_n(w_t/t) + \sum_{i=n-t}^{n-1} \alpha^i(w_t/t) \geq 0, \text{ where } 1 \leq t \leq n \leq s; \end{aligned} \tag{9}$$

$$\begin{aligned} &\kappa\sigma_\kappa(\sigma_n(y_s - w_t/t)) + s^{-1}z_\kappa + \sum_{i=0}^{n-1}\alpha^i y_s + \sum_{i=\kappa-t-n}^{k-1} \alpha^i(\sigma_n(w_t/t)) \\ &\geq w_\kappa - (\kappa - t - n)\sigma_{\kappa-t-n}(\sigma_n(w_t/t)) \geq 0, \text{ where } 1 \leq 2t \leq 2n < \kappa \leq s. \end{aligned} \tag{10}$$

Here $\sigma_k(x) = \sigma_k(\alpha, x)$.

Proof: Inequalities (7) and (8) follow from positiveness of the operator w_t and (9), (10) from the inequalities (7), (8) and (4).

LEMMA 1.8 ([3]): *Let $x \in M$. Then*

$$||\sigma_\kappa(x - \sigma_n(x))||_\infty \leq 2\frac{n}{\kappa}||x||_\infty \quad \text{where } \kappa > 2n.$$

We shall denote $\lim_{\kappa\to\infty} \sigma_\kappa(x_n)$ in the norm $||\cdot||_i$ by $\hat{x}_n$. The existence of the limit follows by Lemma 1.5.

LEMMA 1.9: *Let $\{x_n\}_{n=1}^{\infty}$ be a superadditive sequence. Then*

$$\hat{x}_{nm}/n \geq \hat{x}_m \qquad \text{where } n, m = 1, 2, \ldots.$$

Proof: We have from superadditivity

$$x_{nm} \geq x_m + \alpha^m x_m + \cdots + \alpha^{(n-1)m} x_m.$$

Then $\hat{x}_{nm}/n \geq \hat{x}_m$.

LEMMA 1.10 ([18]): *Let $x = x^* \in M$. For every $\epsilon > 0$ there exists $y = y^* \in M, ||y||_\infty \leq 3||x||_\infty$ such that $||x - y||_2 < \epsilon, ||\sigma_\kappa(y) - \hat{x}||_\infty \to 0$ as $\kappa \to \infty$, where $\hat{x} = \lim \sigma_\kappa(\alpha, x)$.*

Proof of the a.s. convergence: Let $\{V_n = x_n - n\sigma_n(x_1)\}_{n=1}^{\infty}, \lim_{n\to\infty} \rho(V_{n/n}) = \gamma$. There exists a subsequence $\{V_{t_\ell}\}_{\ell=1}^{\infty}$, such that $\sup_{s \geq t_\ell} ||V_{t_\ell}/t_\ell - V_s/s||_1 \leq 2^{-2\ell}$, where $t_{\ell+1} = m \cdot t_\ell$ for some natural number $m(\ell)$. Then

$$\rho(V_{t_\ell}) \geq \lim_{s\to\infty} ||V_s/s||_1 - \sup_{s \geq t_\ell} ||V_{t_\ell}/t_l - V_s/s||_1 \geq \gamma - 2^{-2\ell}. \tag{11}$$

We construct sequences $\{Y_s\}_{s=1}^{\infty}$ and $\{Z_n\}_{n=1}^{\infty}$ by Lemma 1.3 for $\{V_n\}_{n=1}^{\infty}$. Let

$$n_\ell = \max\{t_\ell, [(\rho(V_{t_\ell}) \cdot 2^{-2\ell})^{-1}] + 1\}, \quad s'_\ell = \max\{n_\ell, [\rho(\sum_{i=1}^{n_l} Z_i) \cdot 2^{-2\ell})^{-1}] + 1\}.$$

Then for $s > s'_\ell \geq t_\ell$ we have

$$\gamma \geq p(Y_s) = \rho(V_s/s) \geq \gamma - 2^{-2\ell};$$
$$||1/n_\ell \sum_{i=n_\ell - t_\ell}^{n_\ell - 1} \alpha^i (V_{t_\ell}/t_\ell)||_1 \leq 2^{-2\ell}; (s \cdot n_\ell)^{-1} \sum_{i=1}^{n_\ell} \rho(Z_i) \leq 2^{-2\ell}.$$

It follows for $s > s'_\ell$ that

$$\begin{aligned} &||\sigma_{n_\ell}(Y_s - V_{t_\ell}/t_\ell)||_1 \\ &\leq ||\sigma_{n_\ell}(Y_s - V_{t_\ell}/t_\ell) + 1/n_\ell \sum_{i=n_\ell - t_\ell}^{n_\ell - 1} \alpha^i (V_{t_\ell}/t_\ell) + 1/(s \cdot n_l) \sum_{i=1}^{n_\ell} Z_i||_1 \\ &\quad + ||1/(s \cdot n_\ell) \sum_{i=1}^{n_\ell} Z_i||_1 + ||1/n_\ell \sum_{i=n_\ell - t_\ell}^{n_\ell - 1} \alpha^i (V_{t_\ell}/t_\ell)||_1 < 5 \cdot 2^{-\ell}. \end{aligned}$$

By Lemma 1.10 there exists a sequence $\{x_{1,\ell}\}_{\ell=1}^{\infty}$ such that

$$||x_{1,\ell}||_\infty \le 3||x_1||_\infty; \quad ||x_1 - x_{1,\ell}||_2 \le 2^{-2\ell}; \quad ||\sigma_\kappa(\alpha, x_{1,\ell}) - \hat{x}_1||_\infty < 2^{-2\ell}$$

when $\kappa \to \infty$, where $\hat{x}_1 = \lim_{\kappa\to\infty} \ \sigma(x_1)$. We choose $\{V'_{t_\ell}\}_{\ell=1}^{\infty}$ such that

$$||V'_{t_\ell}||_\infty \le 3||V_{t_\ell}||_\infty; \quad ||V_{t_\ell} - V'_{t_\ell}||_2 \le 2^{-2\ell}; \quad ||\sigma_\kappa(V'_{t_\ell}/t_\ell) - \hat{V}_{t_\ell}||_\infty \to 0$$

where $\hat{V}_{t_\ell} = \lim_{\kappa\to\infty} \sigma_\kappa(\alpha, V'_{t_\ell}/t_\ell)$. There exists κ'_ℓ such that, for $\kappa \ge \kappa'_\ell$, the next inequalities are correct:

$$||\sigma_\kappa(V'_{t_\ell}/t_\ell) - \hat{V}_{t_\ell}||_\infty < 2^{-2\ell}; \quad ||\sigma_\kappa(\alpha, x_{1,\ell}) - \hat{x}_1||_\infty < 2^{-2l}.$$

Put

$$\kappa''_\ell = [(2\gamma \cdot n_\ell \cdot 2^{-2\ell})^{-1}] + 1,$$
$$\kappa_1 = \max\{2n_1 + 1; \kappa''_1; k'_1 + n_1 + t_1; [3||V_{t_1}||_\infty \cdot (t_1 + n_1) \cdot 2^{-2})^{-1}] + t_1 + n_1 + 1\},$$

and for $\ell > 1$

$$\begin{aligned} \kappa_\ell = \max\{2n_\ell + 1; \kappa_{\ell-1}; \kappa''_\ell; \kappa'_\ell + n_\ell + t_\ell; \\ [(3||V_{t_\ell}||_\infty \cdot (t_\ell + n_\ell) \cdot 2^{-2\ell})^{-1}] + t_\ell + n_\ell + 1\}, \\ s_\ell = \max\{\kappa + 1, s'_\ell, [(\sum_{\ell=1}^{\kappa_\ell+1} \rho(Z_i))^{-1} \cdot 2^{2\ell}] + 1\}. \end{aligned}$$

Let $1 > \epsilon > 0$ and $\ell_0 = [\log_{1/2} \epsilon] + c_1$ where c_1 is large enough. Then

$$\begin{aligned} &\sum_{\ell \ge \ell_0} 2^\ell (2^\ell ||V'_{t_\ell}/t_l - V_{t_\ell}/t_\ell||_2^2 + \sum_{m=\ell}^{\infty} ||\hat{V}_{t_{m+1}}/t_{m+1} - \hat{V}_{t_m}/t_m||_1 \\ &+ ||\sigma_{n_\ell}(\alpha, (Y_{s_\ell} - V_{t_\ell}/t_\ell))||_1 + 1/s_\ell \sum_{i=1}^{\kappa_{\ell+1}} \rho(Z_i) + 1/\kappa_\ell || \sum_{i=0}^{n_\ell - 1} \alpha^i Y_{S_\ell}||_1 \\ &+ 2^\ell ||x_1 - x_{1,\ell}||_2^2) \le \epsilon/2. \end{aligned}$$

We note that

$$||\hat{V}_{t_{m+1}}/t_{m+1} - \hat{V}_{t_m}/t_m||_1 \le ||V_{t_{m+1}}/t_{m+1} - V_{t_m}/t_m||_1.$$

Let $N > \kappa_{\ell_0}$. We construct a projection E_N by Theorem 1.6 such that

$$(12)\qquad \begin{aligned} &(1-E_N) < \frac{\epsilon}{2}, \quad \|E_n\sigma_p((V'_{t_\ell}/t_\ell - V_{t_\ell}/t_i)^2)E_N\|_\infty < 2^{-2\ell},\\ &\|E_N\sum_{m=\ell}^{N}(\hat V_{t_{m+1}}/t_{m+1} - \hat V_{t_m}/t_m)E_N\|_\infty \le 2^{-\ell};\\ &\|E_N\sigma_p((x_1 - x_{1,\ell})^2)E_N\|_\infty \le 2^{-2\ell};\\ &\|E_N\sigma_p(\sigma_{n_\ell}(Y_{\sigma_\ell} - V_{t_\ell}/t_\ell))E_N\|_\infty \le 2^{-\ell};\\ &\|E_N(1/s_\ell\sum_{i=1}^{\kappa_{\ell+1}} Z_i)E_N\|_\infty \le 2^{-\ell}, \quad \|E_N(1/\kappa_\ell\sum_{i=0}^{n_{\ell-1}}\alpha^i Y_{s_\ell})E_N\|_\infty \le 2^{-\ell}; \end{aligned}$$

$\ell = \ell_0, \dots, N; \quad p = 1, 2, \dots, N.$

Let F be a weak accumulation point for $\{E_n\}_{N\ge\kappa_{\ell_0}}$ and let $F = \int_0^1 \lambda dF_\lambda$ be a spectral decomposition for $F, E = \int_{1/2}^1 dF_\lambda$.

Then $E \le 2F$; $(1 - E) \le 2\cdot\epsilon/2$. By the inequality

$$\lim_N \sup_{n\ge N}\|E_N x E_N\|_\infty \le \delta$$

it follows for positive $X \in M$ that

$$\|EXE\|_\infty = \|X^{1/2}EX^{1/2}\|_\infty \le 2\|X^{1/2}FX^{1/2}\|_\infty \le 2\delta.$$

For $\kappa_\ell \le \kappa \le \kappa_{\ell+1}$ we have:

$$(13)\qquad \begin{aligned} \|E(x_\kappa/\kappa - x_0)E\|_\infty &\le \|E(x_\kappa/k - \sigma_\kappa(x_1) - x_0)E\|_\infty + \|E(\sigma_\kappa(x_1) - \hat x_1)E\|_\infty\\ &\le \|E(V_\kappa/\kappa - \frac{\kappa - t_\ell - n_\ell}{\kappa}\sigma_{k-t_\ell-n_\ell}(\sigma_{n_\ell}(V_{t_\ell}/t_\ell)))E\|_\infty\\ &\quad + \|E\frac{t_\ell - n_\ell}{\kappa}\sigma_{\kappa-t_\ell-n_\ell}(\sigma_{n_\ell}(V_{t_\ell}/t_\ell))E\|_\infty\\ &\quad + \|E\sigma_\kappa(x_1 - x_{1,\ell})E\|_\infty + \|E(\sigma_\kappa(x_{1,\ell}) - \hat x_1)E\|_\infty\\ &\quad + \|E(\sigma_{\kappa-t_\ell-n_\ell}(\sigma_{n_\ell}(V_{t_\ell}/t_\ell)) - \hat x_0)E\|_\infty. \end{aligned}$$

By inequality (10) the first term is not more than

$$(14)\qquad \begin{aligned} &\|E(\sigma_\kappa(\sigma_{n_\ell}(Y_{s_\ell} - V_{t_\ell}/t_\ell)) + (\kappa\cdot s_\ell)^{-1}Z_\ell + k_\ell^{-1}\sum_{i=0}^{n_\ell - 1}\alpha^i Y_{s_\ell}\\ &\quad + \kappa^{-1}\sum_{i=\kappa-t_\ell-n_\ell}^{\kappa-1}\alpha^i(\sigma_{n_\ell}(V_{t_\ell}/t_\ell)))E\|_\infty. \end{aligned}$$

From the next estimates

$$\sigma_\kappa((V'_{t_\ell}/t_\ell - V_{t_\ell}/t_\ell)^2) \geq (\sigma_\kappa(V'_{t_\ell}/t_\ell - V_{t_\ell}/t_\ell))^2$$

and

$$\begin{aligned}
0 &\leq k^{-1} \sum_{i=\kappa-t_\ell-n_\ell}^{\kappa-1} \alpha^i(\sigma_{n_\ell}(V_{t_\ell}/t_\ell)) \\
&= \sigma_\kappa(V_{t_\ell}/t_\ell)) - \frac{\kappa - t_\ell - n_\ell}{\kappa}\sigma_{\kappa-t_\ell-n_\ell}(\sigma_{n_\ell}(V_{t_\ell}/t_\ell)) \\
&\leq \sigma_\kappa(\sigma_{n_\ell}(V_{t_\ell}/t_\ell) - V_{t_\ell}/t_\ell) - -\frac{\kappa - t_\ell - n_\ell}{\kappa}\sigma_{\kappa-t_\ell-n_\ell}(\sigma_{n_\ell}(V_{t_\ell}/t_\ell) - V_{t_\ell}/t_\ell) \\
&\quad + \sigma_\kappa(V_{t_\ell}/t_\ell - V'_{t_\ell}/t_\ell) - \frac{\kappa - t_\ell - n_\ell}{\kappa}\sigma_{\kappa-t_\ell-n_\ell}(\sigma_{n_\ell}(V_{t_\ell}/t_\ell - V'_{t_\ell}/t_\ell)) \\
&\quad + \sigma_\kappa(V'_{t_\ell}/t_\ell) - \hat{V}_{t_\ell} - \frac{\kappa - t_\ell - n_\ell}{\kappa}(\sigma_{\kappa-t_\ell-n_\ell}(V'_{t_\ell}/t_\ell) - \hat{V}_{t_\ell}) + \frac{t_\ell + n_\ell}{\kappa}\hat{V}_{t_\ell} \\
&\leq 2n_\ell(1/\kappa + 1/(\kappa - n_\ell - t_\ell))\|V_{t_\ell}\|_\infty + 2 \cdot 2^{-2\ell} + \frac{t_\ell + n_\ell}{\kappa}\|\hat{V}_{t_\ell}/t_\ell\|_\infty \\
&\quad + \sigma_\kappa(V_{t_\ell}/t_\ell - V'_{t_\ell}/t_\ell) - \frac{\kappa - t_\ell - n_\ell}{\kappa}\sigma_{\kappa-t_\ell-n_\ell}(V_{t_\ell}/t_\ell - V'_{t_\ell}/t_\ell),
\end{aligned}$$

it follows that

$$\|E(\kappa^{-1} \sum_{i=\kappa-t_\ell-n_\ell}^{k-1} \alpha^i(\sigma_{n_\ell}(V_{t_\ell}/t_\ell)))E\| \leq C_2 \cdot 2^{-\ell}, \tag{15}$$

where C_2 is sufficiently large. It follows from inequalities (12) and (15) that the value of (14) is not more than $C'_2 \cdot 2^{-\ell}$. The second and fourth terms are not more than $2^{-\ell}$. It follows by inequality (12) that the third term is not more than $2 \cdot 2^{-\ell}$. Let us estimate the fifth term, which is not more than

$$\begin{aligned}
&\|E(\sigma_{\kappa-t_\ell-n_\ell}(\sigma_{n_\ell}(V_{t_\ell}/t_\ell) - V_{t_\ell}))E\|_\infty + \|E\sigma_{\kappa-t_\ell-n_\ell}(V_{t_\ell} - V'_{t_\ell})E\|_\infty \\
&+ \|E(\sigma_{\kappa-t_\ell-n_\ell}(V'_{t_\ell}) - \hat{V}_{t_\ell})E\|_\infty + \|E(\hat{V}_{t_\ell} - \hat{x}_0)E\|_\infty \\
&\leq \frac{2n_\ell}{\kappa - t_\ell - n_\ell}\|V_{t_\ell}/t_\ell\|_\infty + 2 \cdot 2^{-\ell} + \|E\sum_{s\geq\ell}(\hat{V}_{t_{s+1}}/t_{s+1} - \hat{V}_{t_s}/t_s)E\|_\infty \leq 6 \cdot 2^{-\ell}.
\end{aligned}$$

We used inequality (10) and Lemma 1.9. It follows that $\hat{V}_{t_s} \uparrow \hat{x}_0$. Thus the value of (13) is not more than $c_3 \cdot 2^{-\ell}$ for some constant c_3 or

$$\|E(x_\kappa/\kappa - \hat{x}_1 + \hat{x}_0)E\|_\infty \to 0 \quad \text{as } \kappa \to \infty. \qquad \blacksquare$$

A further extension of Theorem 1.1 to the case of connected amenable locally compact groups may be found in [11].

2. Convergence of Supermartingales

Let $M, M', M_*, \xi_0, ||\cdot||$ be as in section 1. Let $\{M_n\}_{n=1}^{\infty}$ be an increasing sequence of von Neumann subalgebras of M and $M_0 = (\cup_{n=1}^{\infty} M_n)''$, where $(\cup_{n=1}^{\infty} M_n)''$ is the bicommutant of $\cup_{n=1}^{\infty} M_n$. Suppose also that there exists a conditional expectation $\varphi_n : M \to M_n$ with respect to ρ for $n = 1, 2, \ldots$. Let φ_n^*: $M_* \to (M_n)_* \subset M_*$, where $(M_n)_*$ is the predual of M_n and $(\varphi_n^*(\omega))(x) = \omega(\varphi_n(x))$ where $\omega \in M_*, x \in M$. It follows from the definition of φ_n^* that

$$\varphi_n^* \cdot \varphi_m^* = \varphi_{\min\{n,m\}}^*, \qquad \text{where } n, m = 1, 2, \ldots.$$

A sequence $\{\xi_n\}_{n=1}^{\infty}$ of linear Hermitian normal functionals on M is called a supermartingale if the following conditions are satisfied:

(1) $\sup_{n\geq 1} ||\xi_n||_1 < \infty$,

(2) $\varphi_n^*(\xi_n) = \xi_n, \cdot \varphi_n^*(\xi_{n+1}) \geq \xi_n$ for $n = 1, 2, \ldots$.

A set $\{\xi_i\}_{i\in I}$ of linear Hermitian normal functionals is called absolutely continuous if for any decreasing to zero sequence $\{p_n\}_{n=1}^{\infty}$ of projections from M

$$\sup_{i\in I} |\xi_i(p_n)| \to 0 \qquad \text{when } n \to \infty.$$

THEOREM 2.1: *Let $\{\xi_n\}_{n=1}^{\infty}$ be a supermartingale. The following statements are all equivalent:*

(i) *the set $\{\xi_n\}_{n=1}^{\infty}$ is absolutely continuous;*

(ii) *the sequence $\{\xi_n\}_{n=1}^{\infty}$ converges in the norm $||\cdot||_1$ to $\xi_0 \in (M_0)_*$;*

(iii) *the sequence $\{\xi_n\}_{n=1}^{\infty}$ converges in the $\sigma(M^*, M)$ topology.*

If one of the conditions (i) – (iii) holds, then for every $\epsilon > 0$ there exists a projection $E \in M$ such that $(1 - E) < \epsilon$ and

$$\sup_{\substack{x\geq 0 \\ x\in EME}} \left(|(\xi_n - \xi_0)(x)| \cdot \rho(x)^{-1}\right) \to 0 \qquad \textit{when } n \to \infty$$

if we regard $0 \cdot (\infty) = 0$.

COROLLARY: *Let $\{x_n\}_{n=1}^{\infty} \subset M$ be a sequence of selfadjoint operators such that*

$$\varphi_n(x_n) = x_n, \qquad \varphi_n(x_{n+1}) \geq x_n.$$

If $\sup ||x_n||_\infty < +\infty$, then the sequence x_n converges a.s. in M.

To prove Theorem 2.1, we need some preparation. Let φ_n' be the mapping constructed by Lemma 1.2 from the conditional expectation φ_n.

LEMMA 2.2: *The mapping φ'_n is a conditional expectation on some von Neumann subalgebra N_n with respect to ω_1 and*

$$N_n \subset N_{n+1}, \quad \text{where } n = 1, 2, \ldots.$$

Proof: It follows by Lemma 1.2(ii) that φ'_n is a projection, $\varphi'_n(1) = 1$ and $\omega_1(\varphi'_n(a)) = \omega_1(a)$ for all $a \in M'$. Let P_n be the orthogonal projection on $R_n = \{M_n\xi_0\}^-$. Then $\varphi_n(A)R_n \subset R_n$ for $A \in M$ and $\varphi'_n(B)\xi = P_n(B\xi)$ for $B \in M'$ since

$$\begin{aligned}
&\varphi_n(A)\varphi_n(X)\xi_0 = \varphi_n(A\varphi_n(X))\xi_0 = P_n(A\varphi_n(X)\xi_0),\\
&(X\xi_0, \varphi'_n(B)Y\xi_0) = (\varphi'_n(B^*)Y^*X\xi_0, \xi_0) = (Y^*\varphi_n(X)\xi_0, B\xi_0)\\
&\quad = (P_nX\xi_0, BY\xi_0) = (X\xi_0, P_nBY\xi_0),
\end{aligned}$$

where $X \in M$, $Y \in M_n$. Further $\varphi'_n(B)\varphi'_n(A)\xi_0 = P_n(B \cdot \varphi'_n(A)\xi_0)$, and from $\varphi'_n(B)\xi = P_n(B\xi)$ it follows that $\varphi'_n(B \cdot \varphi'_n(A)) = \varphi'_n(B) \cdot \varphi'_n(A)$ where $A, B \in M'$. It follows by normality of φ'_n that $\varphi'_n(M') = \ker(I - \varphi'_n)$ is weakly closed, i.e. $N_n = \varphi'_n(M')$ is a von Neumann algebra. From the Cauchy–Schwarz inequality it follows that φ'_n is a projection of norm one, hence φ'_n is a conditional expectation from M' on the von Neumann subalgebra N_n with respect to ω_1. It is easily seen that $N_n \subset N_{n+1}$, where $n = 1, 2, \ldots$. ∎

LEMMA 2.3: *Let ξ be a linear normal Hermitian functional on M. Then*

$$\|\varphi^*_n(\xi) - \varphi^*_0(\xi)\|_1 \to 0 \quad \text{when } n \to \infty,$$

where φ_0 is a conditional expectation M on M_0 with respect to ρ.

Proof: For every $b = b^* \in M'$ we have $\varphi'_n(w_b) = w_{\varphi'_n(b)}$. It is known that $\varphi'_n(b)$ converges in $\|\cdot\|_2$ to $\tilde{\varphi}_0(b)$ where $\tilde{\varphi}_0$ is the conditional expectation on $(\cup_{n\geq 1} M'_n)''[I]$. Since $x = x^* \in M$ we have $\|\varphi_n(x) - \varphi_0(x)\|_2 \to 0$ when $n \to \infty$ and $\phi'_n(b) \to \varphi'_0(b)$ weakly for $b \in M'$, i.e. $\varphi'_0 = \tilde{\varphi}_0$. Since $\|b\|_1 \leq \|b\|_2$ for selfadjoint $b \in M'$ it follows that

$$\lim_{n\to\infty} \|\varphi^*_n(\omega_b) - \varphi^*_0(\omega_b)\|_1 = \lim_{n\to\infty} \|\varphi'_n(b) - \varphi'_0(b)\|_1 = 0. \quad ∎$$

THEOREM 2.4: *Let $\{\xi_n\}_{n=1}^{\infty} \subset M_*$ be a sequence of Hermitian functionals and let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence of positive numbers with $\varepsilon_n \leq 1$ and $\Sigma_{n=1}^{\infty}\varepsilon_n^{-1}\|\xi_n\|_1 < 1/2$. There exists a projection $E \in M_0$ such that*

$$\rho(E) \geq 1 - 2\sum_{n=1}^{\infty}\varepsilon_n^{-1}\|\xi_n\|_1$$

and $|\xi_n(\varphi_k(x))| < \varepsilon_n\rho(x)$ for $x \in EME$, $x \geq 0$; $n, k = 1, 2, \cdots$.

Note that it is essential in the proof of this theorem that the sequence of subalgebras $\{M_n\}$ increases. We also note that in this theorem ξ_n is not positive.

LEMMA 2.5: *There exists a set Ω which is $\|\cdot\|_1$ - dense in the space of all linear Hermitian normal functionals, such that for $\xi \in \Omega$ we have*

$$\sup_{\substack{x\in M\\ x\geq 0}} |(\xi(\varphi_N(x)) - \xi(\varphi_0(x)))\cdot(\rho(x))^{-1}| \to 0.$$

Proof: Let $\delta > 0$. There exists by Lemma 2.3 a number N_0 such that

$$\|\varphi_n^*(\omega_b) - \varphi_0^*(\omega_b)\|_1 < \delta \quad \text{for } n \geq N_0.$$

Let

$$\hat{\omega} = \omega_b - \varphi_0^*(\omega_b) + \sum_{n=1}^{\infty} 2^{-n}\varphi_{N_0+n}^*(\omega_b).$$

Then $\|\hat{\omega} - \omega_\ell\|_1 \leq \delta$. Since $|(bx\xi_0, \xi_0)| \leq \|b\|_\infty\rho(x)$ for $b = b^* \in M', x \geq 0, x \in M$, it follows that

$$\sup_{\substack{x\geq 0\\ x\in M}} |(\varphi_N^*(\hat{\omega}) - \varphi_0^*(\hat{\omega}))(x)|(\rho(x))^{-1} \leq \|b\|_\infty 2^{-N+N_0} \to 0 \quad \text{when} \quad N \to \infty. \qquad \blacksquare$$

Proof of Theorem 2.1: The sequence $\{\xi_n\}_{n=1}^{\infty}$ is bounded in M^*. Hence it is relatively $\sigma(M^*, M)$ compact. Let $\{\xi_{n_\gamma}\}_{\gamma\in\Gamma}$ be a set $\sigma(M^*, M)$ — converging to $\xi_0 \in M^*$. Since $\varphi_0^*(\xi_{n_\gamma}) = \xi_{n_\gamma}$ it follows that $\varphi_0^*(\xi_0) = \xi_0$ and ξ_0 is a Hermitian functional. It follows from the fact that $\varphi_\ell^*(\xi_{n_\gamma}) \geq \xi_e$ when $n_\gamma \geq \ell$ that

$$\varphi_\ell^*(\xi_0) \geq \xi_\ell \qquad \text{for } \ell = 1, 2, \cdots. \tag{3}$$

It follows from (3) and (2) that

$$0 \le \varphi_n^*(\varphi_{n+1}^*(\xi_0) - \xi_n) \le \varphi_n^*(\xi_0) - \xi_n. \tag{4}$$

Let ξ_0' be some limit point of the set $\{\xi_n\}_{n=1}^{\infty}$ and let $\{\xi_{n_{\gamma'}}\}_{\gamma' \in \Gamma'}$ be a net which converges to ξ_0'. From the inequality

$$0 \le \varphi_\ell^*(\varphi_{n_{\gamma'}}^*(\xi_0) - \xi_{n_{\gamma'}}) = \varphi_\ell^*(\xi_0) - \varphi_\ell^*(\xi_{n_{\gamma'}})$$

it follows that $\varphi_\ell^*(\xi_0) \ge \varphi_\ell^*(\xi_0')$ and also $\varphi_\ell^*(\xi_0) \le \varphi_\ell^*(\xi_0')$ so that

$$\varphi_\ell^*(\xi_0) = \varphi_\ell^*(\xi_0'). \tag{5}$$

Thus

$$\|\varphi_n^*(\xi_0) - \xi_n\|_1 \to 0 \qquad \text{when } n \to \infty \tag{6}$$

because by (4) it follows that

$$\|\varphi_N^*(\xi_0) - \xi_N\|_1 \le \|\varphi_{n_\gamma}^*(\xi_0) - \xi_{n_\gamma}\|_1 = \xi_0(1) - \xi_{n_\gamma}(1)$$

when $N > n_\gamma$. It follows by (4) and (6) that $\varphi_\ell^*(\xi_0) \in (M_\ell)_*$.
(i) $\Rightarrow$ (ii). The sequence $\{\xi_n\}_{n=1}^{\infty}$ is relatively $\sigma(M_*, M)$ compact by ([20] p. 149). Let $\{\xi_{n_\gamma}\}_{\gamma \in \Gamma}$ be a net $\sigma(M_*, M)$ — converging to $\xi_0 \in M_*$. It follows by Lemma 2.3 and (6) that

$$\|\xi_n - \xi_0\| = \|\xi_n - \varphi_n^*(\xi_0)\|_1 + \|\varphi_n^*(\xi_0) - \xi_0\|_1 \to 0 \quad \text{when } n \to \infty.$$

(ii) $\Rightarrow$ (iii) This is obvious,
(iii)$\Rightarrow$ (i) This follows by weak sequential completeness of M_* ([22] p. 148). Let ℓ_t be a natural number such that when $n \ge \ell_t$, we have $\|\varphi_n^*(\xi_0) - \xi_n\|_1 < 2^{-2t}$. Let $\omega_t = \sum_{n=\ell_t}^{\infty} (\varphi_n^*(\xi_0) - \xi_n - (\varphi_n^*(\xi_0) - \varphi_n^*(\xi_{n+1})))$.

It follows from the positiveness of the terms in the series and (6) that

$$\|\omega_t\| \le (\varphi_{\ell_t}^*(\xi_0) - \xi_{\ell_t})(1).$$

Besides, for $\ell > \ell_t$ we have

$$\begin{aligned}\varphi_\ell^*(\omega_t) &= \sum_{n=\ell_t}^{\ell-1} ((\varphi_n^*(\xi_i) - \xi_n) - \varphi_n^*(\xi_n) + \varphi_n^*(\xi_{n+1})) + \varphi_\ell^*(\xi_0) - \xi_\ell \ge \varphi_\ell^*(\xi_0) \\ &= \xi_\ell. \end{aligned} \tag{7}$$

Let $\xi_0^{(n)} = \Sigma_{\kappa=1}^{\infty} \xi_{0,\kappa}$ be a decomposition by elements from the set Ω which is constructed in Lemma 2.5, where $\|\xi_{0,\kappa}\|_1 \leq 2^{-2\kappa+1}$ when $\kappa \geq 2$. Let $\varepsilon > 0$ and $\kappa_0 > -\log_{\frac{1}{2}} \varepsilon + 3$. Then

$$\sum_{n=\kappa_0}^{\infty} (2^n \|\omega_n\|_1 + 2 \cdot 2^n \cdot \|\xi_{0,n}\|_1) \leq \frac{\varepsilon}{2}.$$

Let $\delta > 0$, $t'_0 = [-\log_{\frac{1}{2}} \delta] + 2, n_0 = \max\{\ell_{\kappa_0}, t'_0\}$. There exists by Theorem 2.4 a projection $E \in M_0$ such that $\rho(E) \geq 1 - \varepsilon$ and

$$\sup_{\substack{x \in EME \\ x \geq 0}} (\xi_{0,\kappa}(\varphi_n(x)) - \xi_{0,\kappa}(\varphi_0(x))) \cdot (\rho(x))^{-1} \leq 2^{-\kappa},$$

$$\sup_{\substack{x \in EME \\ x \geq 0}} |(\omega_\kappa(x))| \cdot (\rho(x))^{-1} \leq 2^{-\kappa} \qquad \text{for } \kappa \geq \kappa_0 \text{ and } n = 1, 2, \tag{8}$$

$$\begin{aligned} \sup_{\substack{x \in EME \\ x \geq 0}} (|(\xi_n - \xi_0)(x)|(\rho(x))^{-1}) \leq & \sup_{\substack{x \in EME \\ x \geq 0}} (|(\xi_n - \xi_0)(\varphi_n(x))| \cdot (\rho(x))^{-1}) \\ & + \sup_{\substack{x \in EME \\ x \geq 0}} (|(\xi_0(\varphi_n(x)) - \xi_0(x))|(\rho(x))^{-1}). \end{aligned} \tag{9}$$

The second term in (9) is not more than

$$\begin{aligned} & \sup_{\substack{x \in EME \\ x \geq 0}} (|(\sum_{\ell=1}^{n} \xi_{0,\ell})(\varphi_n(x) - \varphi_0(x))|(\rho(x))^{-1}) \\ & \quad + \sum_{\ell=n_0}^{\infty} \sup_{\substack{x \in EME \\ x \geq 0}} (|\xi_{0,\ell}(\varphi_n(x) - \varphi_0(x))|(\rho(x))^{-1}) \\ & \leq \sup_{\substack{x \in EME \\ x \geq 0}} (\sum_{\ell=1}^{n} |(\varphi_n(x) - \varphi_0(x))|(\rho(x))^{-1}) + \frac{\delta}{4}. \end{aligned}$$

It follows from inequalities (7) and (8) that the first term is not more that $\delta/4$. We have by construction that

$$A_n = \sup_{\substack{x \in EME \\ x \geq 0}} (|(\sum_{i=1}^{n_0} \xi_{0,\ell})(\varphi_n(x) - \varphi_0(x))|(\rho(x))^{-1}) \to 0,$$

when $n \to \infty$. There exists $n_1 > n_0$ such that $A_n \leq \delta/2$ when $n > n_1$. Hence the value of (9) is not more than δ, when $n > n_1$. Thus the statement of Theorem 2.1 holds. ∎

3. Majorant Ergodic Theorem

Let M be a von Neumann algebra with a faithful normal tracial state τ, and let α be a linear positive normal mapping $M \to M$ such that

$$\alpha(1) \le 1; \quad \tau(\alpha(x)) \le \tau(x) \tag{1}$$

for all $x \in M_+$. The mapping α has a unique extension to a linear continuous operator (which we shall also denote by α) from the space $L_p(M,\tau)$ into $L_p(M,\tau)$, where $L_p(M,\tau)$ is the space of τ-integrable operators affiliated to M ([19], [24]). Let $\sigma_n(\alpha, A)$ be as earlier. It follows by reflexivity of $L_p(M,\tau)$ [24], that $\sigma_n(\alpha, A)$ converges to $E(A)$, where E is a projection on the subspace of α-invariant operators in $L_p(M,\tau)(1 < \rho < \infty)$. It is known (see [24]) that $\sigma_n(\alpha, A)$ converges a.s. to $E(A)$. The sequence (multisequence) $\{A_n\}_{n=1}^{\infty}(\{An_1, n_2, \dots, n_m\}_{n_i=1}^{\infty}, i = \overline{1,m})$ is called (0)-convergent to $A_0 \in L_p(m,\tau)$ when $n(n_i, i = \overline{1,m}) \to \infty$ if there exists a decreasing sequence of selfadjoint positive operators $\{B_n\}_{n=1}^{\infty} \in L_p(M,\tau)$ such that $\inf_n B_n = 0, -B_n \le A_n - A_0 \le B_n$ for $n = 1, 2, \dots$

$$(-B_n \le A_{n_1,\dots,n_m} - A_0 \le B_n, \;\; n = \min_{i=\overline{1,m}} n_i).$$

THEOREM 3.1: *For every positive operator $A \in L_{p+\varepsilon}(M,\tau)$ $(1 \le p < \infty, \varepsilon > 0)$ there exists $B \in L_p(M,\tau)$ such that*

$$\|B\|_p \le C_{p,\varepsilon}\|A\|_{p+\varepsilon}; \sigma_n(\alpha, A) \le B, \quad \text{for } n = 1, 2, \dots$$

where $C_{p,\varepsilon}$ is some constant.

Proof: Let $C \in L_p(M,\tau)$, $C \ge 0$. By Theorem 1.2 [6] for every $\lambda > 0$ there exists a projection $q \in M$ such that

$$\tau(1-q) < 2\lambda^{-1}\tau(C),$$

$$q\sigma_\ell(\alpha, C)q \in M; \|q\sigma_\ell(\alpha, C)q\|_\infty \le \lambda, \;\; \text{for } \ell = 1, 2, \dots.$$

Denote $A \cdot \chi_{[\gamma,+\infty)}(A)$ by $A(\gamma)$, where $\chi_\varepsilon(x)$ is the indicator function of the set ε. There exists a projection $q_1 \in M$ for $\lambda = \gamma/2$ and $C = A(\gamma/2)$ such that

$$\tau(1-q_1) < 4\gamma^{-1}\tau(A(\gamma/2)); \qquad q_1\sigma_\ell(\alpha, A(\gamma/2))q_1 \in M;$$

$$\|q_1\sigma_\ell(\alpha, A)q_1\|_\infty \le \|q_1\sigma_\ell(\alpha, (A - A(\gamma/2)))q_1\|_\infty + \|q_1\sigma_\ell(\alpha, A(\gamma/2))q_1\|_\infty \le \gamma.$$

Let $\gamma_n = \|A\|_{p+1}e^n$, for $n = 0, 1, \ldots$. There exist projections $q_n \in M$ such that $q_n\sigma_\ell(\alpha, A)q_n \in M$, $\|q_n\sigma_\ell(\alpha, A)q_n\|_\infty \le \gamma_n$, for all $n, \ell = 1, 2, \ldots$ and also

$$\tau(1 - q_n) \le 4\gamma_n^{-1}\tau(A(\gamma_n/2)).$$

Let $g_0 = 1 - q_0$; $g_n = g_{n-1} \wedge q_n$, for $n = 1, 2, \ldots$. Then

$$\tau(g_n) \le \tau(1 - q_n) \le 4\gamma_n \cdot \tau(A(\gamma_n/2)).$$

Let $f_n = g_{n-1} - g_n = g_{n-1} \wedge q_n$, $\quad f_0 = g_0$. Then

$$\gamma_n \le \|q_n\sigma_\ell(\alpha, A(\gamma_n/2))q_n\|_\infty = \|\sigma_\ell(\alpha, A(\gamma_n/2))^{\frac{1}{2}}q_n\|_\infty^2$$

$$\le \|\sigma_\ell(\alpha, A(\gamma_n/2))^{\frac{1}{2}}f_n\|_\infty^2 = \|f_n\sigma_\ell(\alpha, A(\gamma_n/2))f_n\|_\infty, \text{ for all } \ell, n = 1, 2, \cdots.$$

Let $\delta_n = n^2$. Then

$$\sigma_\ell(\alpha, A) \le 2g_0\sigma_\ell(\alpha, A)g_0 + 2f_0\sigma_\ell(\alpha, A)f_0 \le 2\gamma \cdot f_0 + 2g_0\sigma_\ell(\alpha, A)g_0,$$

$$2g_0\sigma_\ell(\alpha, A)g_0 \le (1 + \delta_1^{-1})(1 + \delta_\ell)\gamma_i f_i + \prod_{i=1}^{2}(1 + \delta_\ell^{-1})g_1\sigma_\ell(\alpha, A)g_1,$$

$$\prod_{i=1}^{n}(1 + \delta_i^{-1})g_{n-1}\sigma_\ell(\alpha, A)g_{n-1} \le \prod_{i=1}^{n}(1 + \delta_i^{-1})(1 + \delta_{n+1})\gamma_n \cdot f_n + \prod_{i=1}^{n+1}(1 + \delta_i^{-1})g_n\sigma_\ell(\alpha, A)g_n.$$

Thus

$$\sigma_\ell(\alpha, A) \le 2\gamma_0 f_0 + \sum_{m=1}^{N} \Pi_{i=1}^{m}(1 + \delta_i^{-1})(1 + \delta_{m+1})\gamma_m f_m + \Pi_{i=1}^{N+1}(1 + \delta_i^{-1})g_N\sigma_\ell(\alpha, A)g_N, \quad \text{for } \ell = 1, 2, \cdots.$$

Denote $\Sigma_{i=1}^{N}(1 + \delta_i^{-1})g_N\sigma_\ell(\alpha, A)g_N$ by $B_{N,\ell}$ and the sum of the two terms in (2) by B_N.

Let us show that $B_N \in L_p(M, \tau)$, $\|B_N\|_p \le C_{p,\epsilon}\|A\|_{p+\epsilon}$. We have

$$\|B_N\|_p^p = \tau(2^p\gamma_0^p f_0 + \sum_{m=1}^{N}(\Pi_{m=1}^{N}(1 + \delta_{m+1}))^p\gamma_m^p \cdot f_m)$$

$$\le 2^p\gamma_0^p + 2^{3p}3^{2p}\gamma_0^p\tau(g_0) + \sum_{m=2}^{N}(m + 2)^{2p}e^{pm}\tau(q_m) \cdot \gamma_0^p,$$

$$\tau(q_m) \leq 4\gamma_m^{-1}\tau(A(\gamma_{m-1})) \leq 4\gamma_m^{-1} \sum_{i=m-1}^{\infty} \gamma_{i+1}\tau(\chi_{[\gamma_i,\gamma_{i+1}]}(A)),$$

$$\sum_{i=1}^{\infty} \gamma_{i+1}\tau(\chi_{[\gamma_i,\gamma_{i+1}]}(A)) \leq e\tau\ (A)) \in L_{p+\varepsilon}(M,\tau), \qquad \text{when } p \geq 1.$$

It follows that

$$\begin{aligned} \|B_N\|_p^p &\leq 2^p\gamma_0^p + 3^{5p}\gamma_0^p 4 \\ &\quad + 4\sum_{i=1}^{\infty}\left(\sum_{m=2}^{\min\{N,i+1\}} (m+2)^{2p}\ell^{(p-1)m}\right)\cdot\gamma_0^p\cdot e^i\tau(\chi_{[\gamma_{i-1},\gamma_i]}(A)) \\ &\leq C_1\gamma_0^p + e^{p+1}4\sum_{i=1}^{\infty}(i+4)^{2p+1}e^{p(i-1)}\cdot\gamma_0^p\cdot\tau(\chi_{[\gamma_{i-1},\gamma_i]}(A)) = D, \end{aligned}$$

since

$$\sum_{m=2}^{\min\{N,i+1\}} (m+2)^{2p} \leq (i+4)^{2p+1} \leq \frac{C_2}{4}e^{\varepsilon(i-1)} \quad \text{for } i = 1,2,\ldots;$$

then D is not more than the next value

$$C_1\gamma_0^p + C_3(1/\gamma_0)^\varepsilon \sum_{i=1}^{\infty} \gamma_0^{p+\varepsilon} e^{(p+\varepsilon)(i-1)}\tau(\chi_{[\gamma_{i-1};\gamma_i]}(A))$$

$$\leq C_1\gamma_0^p + C_4\|A\|_{p+\varepsilon}^p = C_{p,\varepsilon}^p\|A\|_{p+\varepsilon}^p.$$

Then $\|B_N\|_p \leq C_{p,\varepsilon}\|A\|_{p+\varepsilon}$ where $C_{p,\varepsilon}$ does not depend on N and A. The sequence $\{B_N\}_{N=1}^{\infty}$ is increasing and norm bounded in $L_p(M,\tau)$. It follows that there exist

$$B \in L_p(M,\tau), \quad B \geq 0, \quad \|B\|_p \leq C_{p,\varepsilon}\|A\|_{p+\varepsilon}, \quad B = \lim_{N\to\infty} B_N.$$

We have $B_{N,\ell} \leq \ell^2 g_N\sigma_\ell(\alpha,A)g_N$.

Since $\|AB\|_p \leq \|A\|_q\cdot\|B\|_r, 1/q+1/r = 1/p$ it follows that

$$\|g_N\sigma_\ell(\alpha,A)g_N\|_p \leq \|g_N\sigma_\ell(\alpha,A)\|_{p+\varepsilon}\|g_N\|_{s'} \leq \|\sigma_\ell(\alpha,A)\|_{p+\varepsilon}\|g_N\|_{s'},$$

where $s' = (-(p+\varepsilon)^{-1}+p^{-1})^{-1}$. It follows that $\|B_{N,\ell}\|_p \to 0$ when $N\to\infty$. Besides $\sigma_\ell(\alpha, A l B_N + B_{N,\ell} \leq B + B_{N,\ell}$. When $N\to\infty$ we have $\sigma_\ell(\alpha,A) \leq B$. ∎

THEOREM 3.2: *Let $A \in L_{p+\varepsilon}(M,\tau)$ where $1 \le p < \infty, \varepsilon > 0$ and $A = A^*$. Then $\sigma_n(\alpha, A)$ is (0)-convergent in $L_p(M,\tau)$.*

Proof: Assume that $A \ge 0$. Let $A_m = \chi_{[\beta_m,\infty)}(A) \cdot A$ where β_m is a positive number, $\beta_m \uparrow \infty$, such that $\|A_m\|_{p+\varepsilon} \le 2^{-m-3}$ when $m \ge 2$. Let

$$C_m = A - A_m - E(A - A_m) \in M.$$

Then

$$\sigma_n(\alpha, C_m) = \sigma_n(\alpha, (C_m - \sigma_k(\alpha, C_m)) + \sigma_n(\alpha, \sigma_k(\alpha, C_m)),$$

$$\frac{-2\kappa}{n}||C_m||_\infty - \sigma_n(\alpha, \sigma_\kappa(C_m)) \le \sigma_n(\alpha, C_m) \le \frac{2\kappa}{n}||\sigma_m||_\infty + \sigma_n(\alpha, \sigma_\kappa(\alpha, C_m)).$$

Since $\|\sigma_\kappa(\alpha, C_m)\|_{p+\varepsilon} \to 0$, there exists a number $\kappa(m)$ such that

$$\|\sigma_\kappa(\alpha, C_m)\|_{p+\varepsilon} < 2^{-m-3}, \quad \text{when } k > k(m).$$

By Theorem 3.1 there exists $B''_m \in L_p(M,\tau)$ such that

$$B''_m \ge 0; \quad \|B''_m\|_p \le C_{p,\varepsilon}2^{-m-3}; \quad -2^{-(m+3)} - B''_m \le \sigma_n(\alpha, C_m) \le 2^{-(m+3)} + B''_m,$$

when

$$n > n(m) = [2^{m+2}\kappa(m)\|C_m\|_\infty^{-1}] + 1.$$

By Theorem 3.1 there exists $B'_m \in L_p(M,\tau)$ such that $B'_m \ge 0$,

$$-B'_m \le \sigma_n(\alpha, A_m) - E(A_m) \le B'_m, \|B'_m\|_p \le 2 \cdot C_{p,\varepsilon} \cdot 2^{-m-3}.$$

When $n > n(m)$ we have

$$-2^{-(m+3)} - B''_m - B'_m \le \sigma_n(\alpha, A) - E(A) \le 2^{-(m+3)} + B''_m + B'_m.$$

Let

$$B_\ell = \sum_{i=\ell}^{n(1)} \sigma_i(\alpha, A) + B_{n(1)} \quad \text{when } \ell < n(1)$$

and

$$B_\ell = \sum_{m=\kappa}^{\infty} (2^{-(m+3)} + B''_m + B'_m) \quad \text{when } n(\kappa) \le \ell < n(\kappa+1).$$

Then

$$B_\ell \ge 0, \quad B_\ell \in L_p(M,\tau), \quad B_\ell \downarrow 0, \quad -B_\ell \le \sigma_\ell(\alpha, A) - (A) \le B_\ell. \quad ∎$$

Let M, τ be as earlier and let $\alpha_i, i = \overline{1,m}$ be a linear positive mapping from M into M satisfying condition (1); E_i is a projection of an α_i-invariant subspace in $L_p(M,\tau)$.

Theorem 3.3: *Let $A \in L_{p+\varepsilon}(M,\tau)$ $(1 \le p < \infty, \varepsilon > 0)$, $A = A^*$. Then*

$$\frac{1}{n_1}\frac{1}{n_2}\cdots\frac{1}{n_m}\sum_{i_1=1}^{n_1-1}\cdots\sum_{i_m=1}^{n_m-1}\alpha_1^{i_1}\alpha_2^{i_2}\cdots\alpha_m^{i_m}(A)$$

is (0)-convergent to $E_1\cdots E_m(A)$ in $L_p(M,\tau)$ when $\ell \to \infty$.

Proof: Let us show by induction that for every selfadjoint operator

$$A \in L_{p+\varepsilon}(M,\tau)$$

there exists a sequence $\{B_\ell^{(m)}\}_{\ell=1}^\infty \subset L_{p+\varepsilon/2^m}(M,\tau)$ of positive operators such that $B_\ell^{(m)} \downarrow 0$ when $\ell \to \infty$ and

$$-B_\ell^{(m)} \le \sigma_{\ell_m}(\alpha, \sigma_{\ell_{m-1}}(\alpha_{m-1}, \cdots \sigma_{\ell_1}(\alpha_1, A))\cdots) - E_m\cdots E_1(A) \le B_\ell^{(m)}$$

for all $(\ell_m, \cdots, \ell_1)$ such that $\min_{1\le i\le m}\{\ell_i\} \ge \ell$.

When $m = 1$ this assertion is Theorem 3.2. Let us prove it for $m = k$. There exists a sequence $\{B_{\ell,1}^{(\kappa-1)}\}_{\ell=1}^\infty \subset L_{p+\varepsilon/2^{n-1}}(M,\tau)$ by induction such that $B_{\ell,1}^{(\kappa-1)} \ge 0$; $B_{\ell,1}^{(\kappa-1)} \downarrow 0$ when $\ell \to \infty$ and

$$-B_{\ell,1}^{(\kappa-1)} \le \frac{1}{n_1}\cdots\frac{1}{n_{k-1}}\sum_{i=1}^{n_1-1}\cdots\sum_{i_{k-1}=1}^{n_k-1}\alpha_1^{i_1}\cdots\alpha_{\kappa-1}^{i_{k-1}}(A) - E_1\cdots E_{k-1}(A) \le B_{\ell,1}^{(\kappa-1)}$$

when $\min_{1\le i\le k-1}\{\ell_i\} \ge \ell$. Note that $B_{\ell,1}^{(k-1)} \to 0$ in $\|\cdot\|_{p+\varepsilon/2^{k-1}}$ norm.

Let $\{B_{\ell_t,1}^{(k-1)}\}_{t=1}^\infty$ be a subsequence such that

$$B_{\ell,1}^{(k-1)} = B_{1,1}^{(k-1)}, \sum_{i=1}^\infty \|B_{\ell_t,1}^{(k-1)}\|_{p+\varepsilon/2^{k-1}} \le 2\cdot\|B_{1,1}^{(k+1)}\|_{p+\varepsilon/2^{k-1}}. \tag{4}$$

By Theorem 3.1 there exists $B_t \in L_{p+\varepsilon/2^k}$ such that

$$\sigma_{\ell_k}(\alpha_k, B_{\ell_t,1}^{(k-1)}) \le B_t; \quad \|B_t\|_{p+\varepsilon/2^k} \le C_k\cdot\|B_{\ell_t,1}^{(k-1)}\|_{p+\varepsilon/2^{k-1}} \quad \text{for } t,\ell = 1,2,\ldots.$$

It follows by (4) that the series $B_{t,1} = \Sigma_{i=t}^\infty B_i$ converges in $L_{p+\varepsilon/2^k}$, $B_{t,1} \downarrow 0$ when $t \to \infty$ and $\sigma_{\ell_\kappa}(\alpha_\kappa, B_{\ell_t,1}^{(\kappa-1)}) \le B_{t,1}$. Let $B_{\ell,2} = B_{t,1}$ for $\ell_t \le \ell \le \ell_{t+1}$. Then

$$\begin{aligned}-B_{\ell,2} &\le -\sigma_{\ell_\kappa}(\alpha_\kappa, B_{\ell_t,1}^{(k-1)})\\ &\le \sigma_{\ell_\kappa}(\alpha_\kappa, \sigma_{\ell_{k-1}}(\alpha_{\kappa-1}\cdots\sigma_{\ell_1}(\alpha_1(A)) - E_{k-1}\cdots E_1(A))\\ &\le \sigma_{\ell_\kappa}(\alpha_\kappa, B_{\ell_t,1}^{(\kappa-1)}) \le B_{\ell,2}, \quad \text{if } \min_{i=1,k}\{\ell_i\} \ge \ell_t.\end{aligned}$$

There exists a sequence $\{B_{\ell,3}\}_{\ell=1}^{\infty} \subset L_{p+\varepsilon/2}$ such that $B_{\ell,3} \downarrow 0$ when $\ell \to \infty$ and

$$-B_{\ell,3} \leq \sigma_{\ell_K}(\alpha_k, \cdots, \sigma_{\ell_1}(\alpha_1, A)) - E_k \cdots E_1(A) \leq B_{\ell,3}$$

when $\ell_k \geq \ell$. Let $B_\ell^{(k)} = B_{\ell,3} + B_{\ell,2}$. Then

$$\{B_\ell^{(k)}\}_{\ell=1}^{\infty} \subset L_{p+\varepsilon/2m}; \quad B_\ell^{(k)} \downarrow 0, \quad \ell \to \infty;$$

$$-B_\ell^{(k)} < \sigma_{\ell_\kappa}(\alpha_\kappa, \cdots (\sigma_{\ell_1}(\alpha_1, A)) \cdots) - E_k \cdots E_1(A) \leq B_\ell^{(k)}$$

for all $(\ell_k, \cdots, \ell_1)$ such that $\min_{1 \leq i \leq k}\{\ell_i\} \geq \ell$. ∎

COROLLARY 3.4: *Let α_i, E_i be as in Theorem 3.3 and*

$$A \in L_{P+\varepsilon}(M, \tau) \quad (1 \leq p < \infty, \quad \varepsilon > 0).$$

Then

$$\sigma_{\ell_k}(\alpha_k, \cdots \sigma_\ell(\alpha_1, A) \cdots) - E_k \cdots E_1(A) \to 0$$

a.s. when $\ell_i, i = \overline{1,k}$ converge to ∞.

References

1. W. Arveson, *Analyticity in operator algebras*, Am. J. Math. **89** (1967), 578–642.
2. C. Barnett, *Supermartingales on semifinite von Neumann algebras*, J. London Math. Soc. **24** (1981), 175–181.
3. J. P. Conze and N. Dang-Ngoc, *Ergodic theorems for non-commutative dynamical systems*, Invent. Math. **46** (1978), 1–15.
4. I. Cuculescu, *Supermartingales on W*-algebras*, Rev. Roumaine Math. Pures Appl. **14** (1969), 759–773.
5. I. Cuculescu, *Martingales on von Neumann algebras*, J. Multivar. Anal. **1** (1971), 17–27.
6. N. Danford and J.T. Schwartz, *Linear Operators*, Vol. I, Interscience, New York, 1958.
7. N. Dang-Ngoc, *Pointwise convergence of martingales in von Neumann algebras*, Israel J. Math. **34** (1979), 273-280.
8. J. Dixmier, *Les algébres d'opérateurs dans l'espace hilbertien* (algebres de von Neumann), Gauthier-Villars, Paris, 1969.

9. M. S. Goldstein, *Theorems of convergence almost everywhere in von Neumann algebras*, J. Operator Theory **6** (1981), 233–311.
10. G. Y. Grabarnik, *Convergence of superadditive sequences and supermartingales on von Neumann algebras*, DAN USSR **11** (1985), 6–8.
11. G. Y. Grabarnik, in press.
12. R. Jajte, *Strong limit theorem in non-commutative probability*, Lecture Notes in Math. **1110**, Springer-Verlag, Berlin, 1985, p. 162.
13. R. V. Kadison, *A generalized Schwarz inequality and algebraic invariants for operator algebras*, Ann. Math. **56** (1952), 494–503.
14. J. F. C. Kingmann, *Subadditive ergodic theory*, Ann. Prob. **1** (1973), 883–909.
15. I. Kovacs and J. Szücs, *Ergodic type theorems in von Neumann algebras*, Acta Sci. Math. (Szeged) **27** (1966), 233–246.
16. U. Krengel, *Ergodic Theorems*, de Greuter, Berlin, 1985, p. 357.
17. E. C. Lance, *Ergodic theorems for convex sets and operator algebras*, Invent. Math. **37** (1976), 201–214.
18. D. Petz, *Ergodic theorems in von Neumann algebras*, Acta Sci. Math. (Szeged) **46** (1983), 329–343.
19. I. E. Segal, *A non-commutative extension of abstract integration*, Ann. Math. **57** (1953), 401–457.
20. Y. G. Sinai and V. V. Anshelevich, *Some problems of non-commutative ergodic theory*, Usp. Mat. Nauk **32** (1976), 157–174.
21. S. Strabila, *Modular Theory in Operator Algebras*, Editura Academiei, Bucuresti, 1981, p. 492.
22. M. Takesaki, *Theory of Operator Algebras*, Springer-Verlag, Berlin, 1949, p. 415.
23. M. Umegaki, *Conditional expectation in an operator algebra III*, Kodai Math. Jem. Rep. **11** (1959), 51–74.
24. F. J. Yeadon, *Ergodic theorems for semifinite von Neumann algebras, I*, J. London Math. Soc. **16** (1977), 326–332.
25. F. J. Yeadon, *Ergodic theorems for semifinite von Neumann algebras II*, Math. Proc. Cambridge Philos. Soc. **88** (1980), 135–147.